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A Horizon Ratio Bound for Inflationary Fluctuations

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We demonstrate that the gravity wave background amplitude implies a robust upper bound on the ratio: $\lambda/H^{-1} \lesssim e^{60}$, where λ is the proper wavelength of fluctuations of interest and H^{-1} is the horizon at the end of inflation. The bound holds as long as the energy density of the universe does not drop faster than radiation subsequent to inflation. This limit implies that the amount of expansion between the time the scales of interest leave the horizon and the end of inflation, denoted by e^N, is also bounded from above, by about e^{60} times a factor that involves an integral over the first slow-roll parameter. In other words, the bound on N is model dependent – we show that for vast classes of slow-roll models, $N \lesssim 67$. The quantities, λ/H^{-1} or N, play an important role in determining the nature of inflationary scalar and tensor fluctuations. We suggest ways to incorporate the above bounds when confronting inflation models with observations. As an example, this bound solidifies the tension between observations of cosmic microwave background (CMB) anisotropies and chaotic inflation with a ϕ^4 potential by closing the escape hatch of large N (< 62).

98.80.Bp; 98.80.Cq; 98.65.Dx

The purpose of this short note is two-fold. First, we briefly review how fluctuations predicted by inflation [1] are related to N, the number of e-folds between the time the scales of interest leave the horizon [2] and the end of inflation (§I). (Note that N is *not* the total number of e-folds of inflation, a generally bigger number.) Second, we derive an upper bound on N which should be used when constraining inflationary models. This is done in two steps:

- we derive a robust, model-independent limit on $\tilde{N}$, defined to be the logarithm of the ratio of the proper wavelength of cosmological modes to the horizon at the end of inflation (§II).
- From this, we infer a model-dependent bound on N (§III).

It is not uncommon to find in the literature a wide variety of assumptions made about N, and we find it timely to point out the importance of this bound, especially in light of improving observations. It should be emphasized that while a fair fraction of our discussion is confined to single-field slow-roll inflation for the sake of simplicity, the constraint on $\tilde{N}$ in §II is quite general, applicable to a much wider variety of inflation models. This leads to a short discussion in §IV where we observe that $\tilde{N}$ might be a better independent variable to adopt instead of N, when solving the inflationary flow equations.

While revision of this paper was under way, a paper by Liddle and Leach [3] appeared which reached very similar conclusions.

I. A BRIEF REVIEW

For large classes of single-field, slow-roll inflationary models, the predictions for scalar and tensor fluctuations can be summarized as follows (to lowest order in slow-roll) [4]:

$$n_s - 1 = \sigma \quad , \quad r = -n_T/2 = \epsilon \tag{1}$$

where n_s is the scalar spectral index, r is the tensor to scalar ratio, and n_T is the tensor spectral index. The equality $r = -n_T/2$ expresses the well-known consistency relation [7].

The slow-roll parameters ϵ and σ are related to derivatives of the Hubble parameter H as a function of inflaton field value ϕ:

$$\epsilon \equiv \frac{m_{\rm pl.}^2}{4\pi}\left(\frac{H'}{H}\right)^2 \quad , \quad \sigma \equiv \frac{m_{\rm pl.}^2}{2\pi}\left[\frac{H''}{H} - 2\left(\frac{H'}{H}\right)^2\right] \tag{2}$$

where the prime denotes differentiation with respect to ϕ. For a wave-mode of interest, equation (1) is to be evaluated at horizon crossing during inflation. This is equivalent to evaluating equation (2) at the corresponding field value $\phi = \phi_*$ (hereafter $*$ is used to denote the time of horizon exit), or, as is commonly done, at the corresponding N:

$$N(\phi_*) \equiv \int_{t_*}^{t_e} dt H = \frac{\sqrt{4\pi}}{m_{\rm pl.}} \left| \int_{\phi_*}^{\phi_e} d\phi/\sqrt{\epsilon} \right| \tag{3}$$

where t is the proper time. Here N is the number of e-folds between the horizon exit of the scale of interest (i.e. t_* or ϕ_*) and the end of inflation (t_e or ϕ_e). The end of inflation is defined to be the time when slow-roll ends.

A hierarchy of flow equations tells us how the slow-roll parameters depend on N [8]:

$$\frac{d\epsilon}{dN} = \epsilon(\sigma + 2\epsilon)\,, \; \frac{d\sigma}{dN} = -5\epsilon\sigma - 12\epsilon^2 + 2(^2\lambda)\,, \tag{4}$$
$$\frac{d(^\ell\lambda)}{dN} = [(\ell-1)\sigma/2 + (\ell-2)\epsilon](^\ell\lambda) + {}^{\ell+1}\lambda$$

where ℓ ranges from 2 to in principle infinity, and ${}^\ell\lambda$'s are the higher order slow-roll parameters. In understanding the dynamics of inflation, it is also useful to remember the equation of motion for ϕ: $\dot\phi = -m_{\rm pl.}^2 H'/(4\pi)$, and the Friedmann equation: $3H^2 = (8\pi/m_{\rm pl.}^2)[V + \dot\phi^2/2]$, where $\dot\phi$ is the derivative of ϕ with respect to proper time, and V is the inflaton potential [9].

As expressed above, it is clear that N plays an important role in determining the properties of observable fluctuations. One can imagine a bound on N provides useful information about the fluctuations, although the precise manner depends on the particular model under consideration. To take a simple example, for chaotic inflation with a ϕ^p potential: $N(\phi_*) + p/4 = 4\pi\phi_*^2/(pm_{\rm pl.}^2)$ (where we have used the fact that $\epsilon = 1$ at the end of inflation), and $\epsilon = p/(p+4N)$, $\sigma = -(2+p)/(2N+p/2)$, leading to (at the lowest order):

$$n_s - 1 = -(2+p)/(2N+p/2) \quad , \quad r = p/(p+4N) \tag{5}$$

The predictions of chaotic inflation then are quite sensitive to the precise value of N, and this dependence holds for many inflationary models [10]. This leads to an important question: what are the constraints on N? The WMAP team [11] fixed N to be 50 and then proceeded to show that their data excluded the ϕ^4 model. Reference [13] pointed out though that N need not be fixed at 50, and loosening this constraint correspondingly loosens the constraints on the ϕ^4 chaotic inflation model. It is not uncommon in the literature to allow N to range up to 70 (e.g. [5]).

II. A MODEL-INDEPENDENT BOUND ON $\tilde N$

First, we derive a bound on a slightly different quantity, which turns out to be more robust. Let us [6] define $e^{\tilde N} \equiv a_e H_e/k$, where k is the comoving wavenumber of interest, a_e is the scale factor and H_e is the Hubble parameter, both at the end of inflation. Hereafter the subscript e refers to the end of inflation. In other words, $e^{\tilde N}$ is the ratio of the physical wavelength (a_e/k) to the Hubble radius (H_e^{-1}) at the end of inflation. It can be calculated backwards from today: there is a symmetry in the evolution of aH/k. During inflation this ratio increases from unity at horizon crossing to $e^{\tilde N}$, and then after inflation it falls back to unity once the scale re-enters the horizon. The bound can be derived by extrapolating backwards from today to get a_e as a function of H_e, so that $\tilde N$ is solely a function of H_e and then arguing that H_e is less than or equal to H_*.

Let us now develop the argument in more detail to make sure we arrive at a conservative bound. Naively, one expects $H_e = H_0\Omega_{r,0}^{1/2} a_e^{-2}$, where $\Omega_{r,0} = 4.2 \times 10^{-5} h^{-2}$ is the radiation density today in units of the critical density, with $h \equiv H_0/$ (100 km/s/Mpc) parametrizing the Hubble constant today. Taking into account changes in the number of relativistic species, as well as the possibility of decoupled degrees of freedom (e.g. neutrinos today), one should use instead $H_e = H_0\Omega_{r,0}^{1/2} a_e^{-2}\left[(g_e/g_0)(g_0^S/g_e^S)^{4/3}\right]^{1/2}$. Here, g is the effective degrees of freedom that relates the energy density ρ to temperature T: $\rho \propto gT^4$, while g^S relates the entropy density s to T: $s \propto g^S T^3$. If g and g^S were identical, then the factor in square brackets would be $(g_0/g_e)^{1/6}$, smaller than $(3.36/100)^{1/6} = 0.57$ since the standard model alone contains more than 100 relativistic degrees of freedom at very high temperatures. The difference between the g's [14] mitigates this to some extent and is somewhat model dependent; a conservative bound follows from setting the coefficient to unity, so $a_e < (H_0/H_e)^{1/2}\Omega_{r,0}^{1/4}$. Thus,

$$e^{\tilde N} = \frac{a_e H_e}{k} < 0.08 \left(\frac{H_e}{H_0}\right)^{1/2} \left(\frac{H_0}{k}\right) h^{-1/2}. \tag{6}$$

Using now the weak assumption that H_*, the Hubble parameter in the early part of inflation when the fluctuation leaves the horizon, is larger than H_e, we arrive at

$$e^{\tilde N} < e^{60.9} \left(\frac{H_*}{10^{15}\mathrm{GeV}}\right)^{1/2} \left(\frac{0.002\,\mathrm{Mpc}^{-1}}{k}\right). \tag{7}$$

Note that $\tilde N$ is a function of scale k. The scale $k = 0.002$ Mpc^{-1} is well-measured by the CMB, so it is a convenient pivot spot [11].

There is one possible loophole in Eq. (6). The end of slow-roll (a_e) is generally earlier than the time when the universe finally completes reheating to become radiation dominated. Equation (6) assumes that this transition is instantaneous, but relaxing this assumption only strengthens the inequality. To see this, for a given H_e, define a quantity $a_e^{\rm eff.}$, which is the scale factor if one were to extrapolate backward from the end of reheating to a time when the Hubble parameter is H_e, as if the universe remains radiation dominated between these two times. With the weak assumption that the true Hubble parameter should fall slower than a^{-2} between these two times, one can see that $a_e < a_e^{\rm eff.}$. Combining this with the relation $a_e^{\rm eff.} < (H_0/H_e)^{1/2}\Omega_{r,0}^{1/4}$ gives us back the inequality in Eq. (6).

The gravity wave amplitude is proportional to H_*. A conservative bound (3σ) from observations of the CMB anisotropies is $H_* < 3.3 \times 10^{14}$ GeV [15]. Hence, Eq. (7) constrains [12]

$$\tilde N < 60 + \ln\left(\frac{0.002\,\mathrm{Mpc}^{-1}}{k}\right). \tag{8}$$

The largest observable scale today corresponds to $k = H_0$, implying the largest possible observationally relevant $\tilde N$ is $62 + \ln(0.7/h)$.

We refer to this limit on $\tilde N$ as the *horizon ratio bound*, as it derives from comparing the horizon today with that at the end of inflation. An important assumption is that the Hubble parameter does not fall faster than a^{-2} after the end of inflation i.e. the energy density does not redshift faster than radiation. If, for instance, there is an extended period of domination by a kinetic-energy-dominated scalar field ($H \propto a^{-3}$), the above bound would be violated. On the other hand, periods of late entropy production or secondary inflation would only serve to strengthen our bound. This caveat aside, our bound is quite general – it is independent of the exact model of inflation.

III. UPPER BOUND(S) ON N

The amount of expansion between horizon exit and the end of inflation is given by $e^N = a_e H_*/k = e^{\tilde{N}} H_*/H_e$. Following equation (6), we see that

$$e^N < 0.08 \left(\frac{H_0}{k}\right) h^{-1/2} \Big[\left(\frac{H_*}{H_e}\right)^{1/2} \left(\frac{H_*}{H_0}\right)^{1/2} \Big] \tag{9}$$

The second term inside the square brackets can be bounded using the gravity wave amplitude as before. The first is the square root of the ratio of the Hubble parameter at exit and at the end of inflation. This ratio can be rewritten using equations (2) and (3): H_*/H_e as a function of N is given by $\exp[\int_0^N \epsilon(N')dN']$ [20]. Hence, we obtain

$$N < 60 + \frac{1}{2}\int_0^N \epsilon(N')dN' + \ln\left(\frac{0.002\,\mathrm{Mpc}^{-1}}{k}\right) \tag{10}$$

The integral over ϵ introduces a dependence on the inflation model to the bound on N. The weakest statement one could make is that $\epsilon < 1$ during inflation, and so the integral has to be less than N, implying a bound on N that is weaker than the one on $\tilde{N}$ by a factor of 2. Imposing the requirement that inflation has to end before nucleosynthesis (temperature ~ 1 MeV) strengthens this bound somewhat to $N < 105 + \ln(0.002\,\mathrm{Mpc}^{-1}/k)$. This is our most general model-independent bound on N.

However, generic single-field slow-roll models (including hybrid models as effective single-field models) likely obey a significantly stronger bound on N. We perform an integration of the flow equations (equation 4) up to the 5th order in slow-roll (i.e. $\ell = 5$), for a million randomly generated models in the slow-roll parameter space, following the prescription of [5]. The trajectories of ϵ can be used to evaluate the integral in equation (10). We solve for the resulting bound on N for each model, whose probability distribution is shown in Fig. 1. It appears there is an upper limit on N:

$$N < 67 + \ln(0.002\,\mathrm{Mpc}^{-1}/k) \tag{11}$$

We do find, however, some evidence for a weak increase in this upper bound as one truncates the slow-roll flow equations at higher orders. We therefore recommend using equation (10) to evaluate the appropriate bound on a case by case basis.

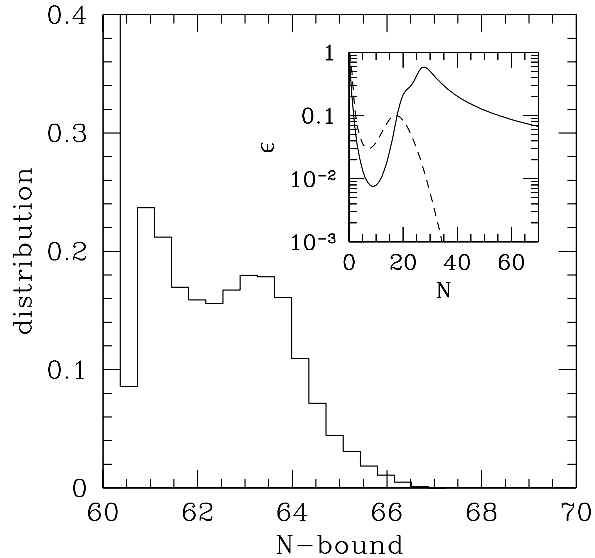

FIG. 1: The probability distribution of N-bound (eq. 10) among a host of Monte Carlo realizations of inflation models. The spike around 60 is largely due to fixed points, models where inflation does not terminate at $\epsilon = 1$, but rather $\epsilon \sim 0$. The inset shows two examples of how ϵ flows with N (i.e. not fixed points).

An instructive example to see why the model-dependent correction to the N-bound is small is chaotic inflation with a ϕ^4 potential. From §I, we know $\epsilon = 1/(1+N)$, and so $\int_0^N \epsilon(N')dN' = \ln(1+N)$. Plugging this into equation (10) implies a bound of $N < 62 + \ln(0.002\,\mathrm{Mpc}^{-1}/k)$. Such a modest N for the ϕ^4 model runs the danger of producing too much spectral tilt and/or too high a tensor to scalar ratio (equation 5). Recently, [21] showed that the combination of WMAP with seven other CMB experiments rules out the ϕ^4 model at 3σ unless N is larger than 66. This, together with our bound, appears to rule out ϕ^4 chaotic inflation. However, we caution that [21] combined different experiments assuming independence.

IV. DISCUSSION

In summary, we have derived a model-independent upper limit of about $e^{\tilde{N}} < e^{60}$ on the ratio of wavelength to horizon size at the end of inflation (equation 8). A corresponding model-dependent upper limit on e^N, which is the amount of expansion between horizon exit and the end of inflation, is given in equation (10). For vast classes of slow-roll models, we find that this gives a bound of $N < 67$.

The discussion so far points to two different ways of implementing the horizon-ratio bound. One is to use equation (10) and evaluate the model-dependent correction on a case by case basis. The other is to bypass the use of N altogether. It can be shown from equations (2) and (3) that

$$(1+\epsilon)\frac{d}{d\tilde{N}} = \frac{d}{dN} \tag{12}$$

This can be used to rewrite the flow equations (4) using $\tilde{N}$ instead of N as the independent variable. The predictions for inflationary fluctuations can therefore be expressed in terms of $\tilde{N}$ in place of N. Our robust bound on $\tilde{N}$ can be implemented directly. We will explore this further in a subsequent paper. This constraint is a useful addition to the host of other constraints emerging from cosmological observations [22].

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[10] For new inflation (i.e. small field) with a potential of the form $V = \Lambda^4[1-(\phi/\mu)^p]$, where $p > 2$, and $\phi \ll \mu$, it can be shown that

$$\epsilon = \left[\frac{2^{2p+2}\pi^p}{p^2(p-2)^{2p-2}}\left(\frac{\mu}{m_{\rm pl.}}\right)^{2p} N^{2-2p}\right]^{1/(p-2)}$$

and $\eta = -N^{-1}(p-1)/(p-2)$, which determine n_s, n_T, and A_T/A_S through equation (1). Hybrid inflation, where one field provides the vacuum energy floor while another rolls, can be treated effectively as a single-field model: $V = \Lambda^4[1+(\phi/\mu)^p]$. In this case, ϵ is the same as in new inflation, while η has the opposite sign. See [19] for a more thorough classification.

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